Task 1

Inequalities for some occasions

Solution

Can we find a quadratic inequality for each region of the Venn diagram?

The regions are defined as follows.

A: The solution set is a subset of $x \le 1$.

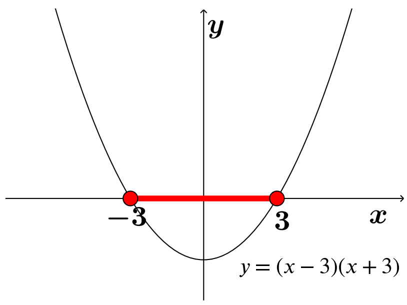
B: The solutions are given by $a \le x \le b$ where a and b are real numbers.

C: The inequality is satisfied by $x = 4$, e.g. $x = 4$ satisfies the inequality $x \ge 2$.

We will solve the given quadratic inequalities so they can be placed in the Venn diagram. We will start with the more straightforward ones.

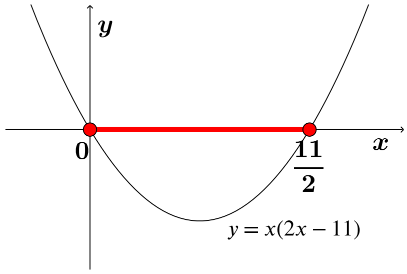
①	$x^2 \le 9$

$x^2 - 9 \le 0$
$(x+3)(x-3) \le 0$
From the graph we can see $-3 \le x \le 3$ are the only real values of x that satisfy the inequality.
Therefore, this inequality satisfies condition B only.



②	$11x \ge 2x^2$

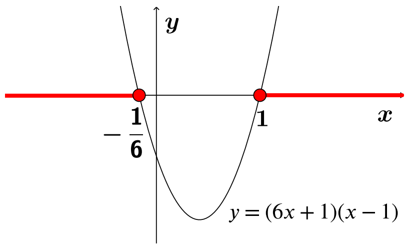
$0 \geq 2x^2 - 11x$
$0 \geq x(2x - 11)$
$0 \leq x \leq \frac{11}{2}$
This inequality satisfies conditions B and C.



Does it matter whether we move the terms to the left or the right hand side of the inequality? What would change if we solved $11x - 2x^2 \geq 0$ instead?

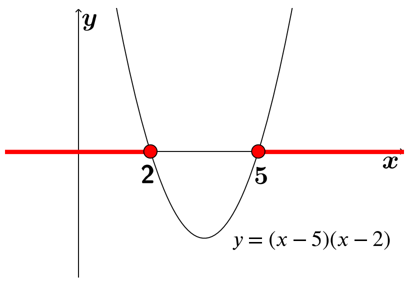
⑦ $6x^2 - 1 \geq 5x$

$6x^2 - 5x - 1 \geq 0$
$(6x + 1)(x - 1) \geq 0$
$-\frac{1}{6} \leq x, \; x \geq 1$
This inequality satisfies condition C.



④ $3x^2 \geq 21x - 30$

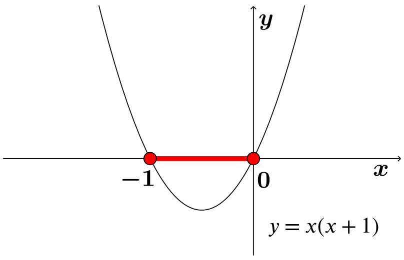
Divide both sides of the inequality by 3
$x^2 \geq 7x - 10$
$x^2 - 7x + 10 \geq 0$
$(x - 5)(x - 2) \geq 0$
$x \geq 5, \; x \leq 2$
This inequality does not satisfy any of the conditions.



Why can we divide both sides by 3? How does the graph of $y = 3x^2 - 21x + 30$ compare to the graph of $y = x^2 - 7x + 10$?

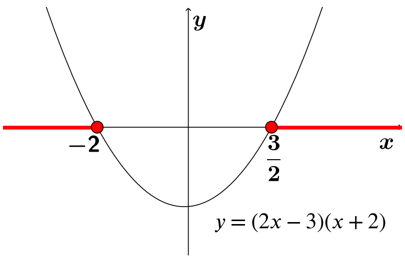
⑤ $x^2 \le -x$

$x^2 + x \le 0$
$x(x+1) \le 0$
$-1 \le x \le 0$
This inequality satisfies conditions A and B.



⑧ $-2x^2 \le x - 6$

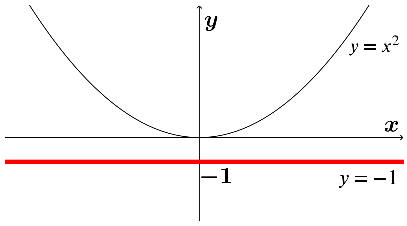
$0 \le 2x^2 + x - 6$
$0 \le (2x-3)(x+2)$
$x \le -2,\ x \ge \frac{3}{2}$
This inequality satisfies C only.



③ $x^2 + 3 \ge 2$

$x^2 \geq -1$

From the graph we can see that this inequality is true for all real values of x. Therefore, this inequality satisfies C only.



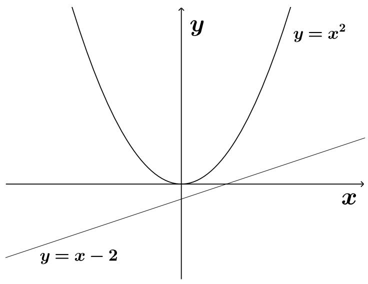
We didn't try to solve this algebraically because of what we know about x^2. What would happen if we did try to solve it algebraically?

⑥ $x^2 \leq x - 2$

$x^2 - x + 2 \leq 0$

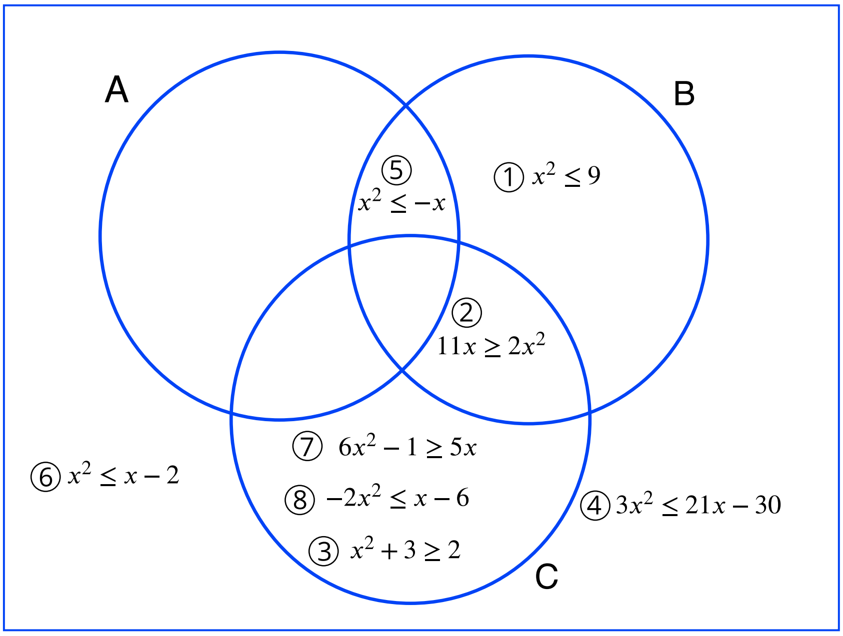
The discriminant of the quadratic $x^2 - x + 2 = 0$ is $(-1)^2 - 4 \times 1 \times 2 = -7$. This means there are no real solutions to the quadratic equation and therefore no real solutions to the quadratic inequality. This can be seen from the graph. For all real values of x, it is impossible for the graph $y = x - 2$ to be greater than the graph $y = x^2$.

Therefore, this inequality does not satisfy any of the conditions.



What other graph might we have sketched to see there are no solutions?

Here are the given inequalities presented in the Venn Diagram.



Are there any inequalities which satisfy conditions A and C or conditions A, B and C?

It is impossible for a number to be less than 1 and equal to 4 simultaneously. Therefore it is impossible to fill these regions.

Are there any inequalities which satisfy only condition A?

One of the ways for the solution set to be a subset of $x \leq 1$ is if it is bounded by two values, e.g. $-3 \leq x \leq 0$, but this would mean the solution set also satisfies condition B.

Another would be if the solution set only has one bound, e.g. $x \leq -4$, but this is impossible for a quadratic inequality.

However, we could consider a case whose solution set is a point, e.g. $(x+1)^2 \leq 0$ has the solution set $x = -1$ which satisfies condition A only.

There is some discussion about this solution, as this can also be written as $-1 \leq x \leq -1$ which then means it also satisfies condition B.

You might have suggested a quadratic inequality with a solution set such as $-3 < x < 0$ (using different inequality signs) and this would indeed satisfy only condition A. However we chose to only use the signs $\leq$ and $\geq$.

Task 2

Careful use of Pythagoras and SOHCAHTOA should give an answer close to **12.07 cm**. Make sure you keep all decimal places in your working out as rounding too early can cause wild variations in the accuracy of your answer.

Task 3

There are a number of ways to solve this Sudoku. The first step is to find equations which are similar and allow you to solve for a variable.

For example, a logical start might be to notice that if:

$m + k + c = 16$ and $m + k + c + g = 23$ then we must have $g = 7$.

Similar reasoning along with the rules of Sudoku yields the following completed puzzle:

8	4	6	9	3	1	5	7	2
5	9	3	4	7	2	8	1	6
1	2	7	6	8	5	4	9	3
3	5	1	2	9	7	6	8	4
7	6	9	5	4	8	2	3	1
4	8	2	1	6	3	7	5	9
9	7	5	3	2	6	1	4	8
6	3	8	7	1	4	9	2	5
2	1	4	8	5	9	3	6	7

Task 4

$$2x^2 = 18$$
$$x^2 = 9$$
$$\boldsymbol{x = \pm 3}$$

$$(2x)^2 = 18$$
$$2x = \pm 3\sqrt{2}$$
$$\boldsymbol{x = \frac{\pm 3\sqrt{2}}{2}}$$

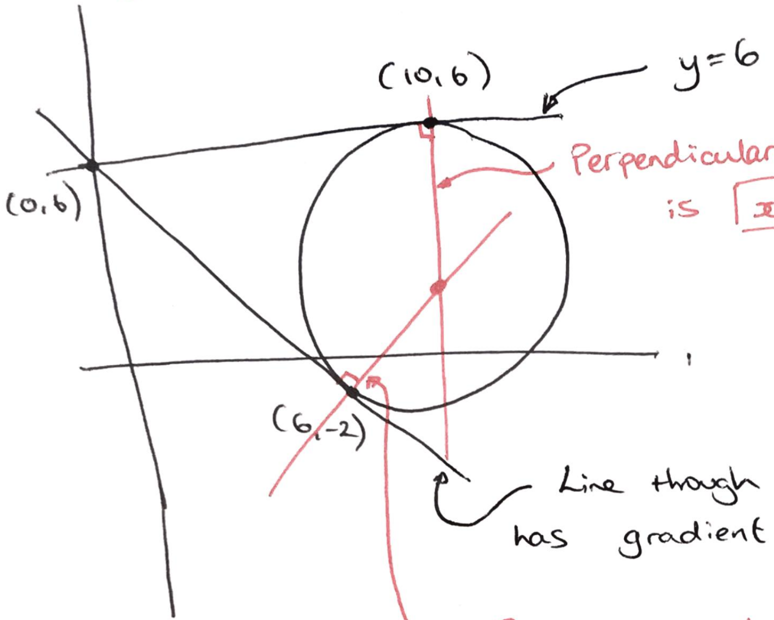
$$2x^2 + 1 = 18$$
$$2x^2 = 17$$
$$x^2 = \frac{17}{2}$$
$$\boldsymbol{x = \pm \frac{\sqrt{17}}{\sqrt{2}}}$$

$$(2x+1)^2 = 18$$
$$2x + 1 = \pm 3\sqrt{2}$$
$$2x = -1 \pm 3\sqrt{2}$$
$$\boldsymbol{x = \frac{-1 \pm 3\sqrt{2}}{2}}$$

$$2(x+1)^2 = 18$$
$$(x+1)^2 = 9$$
$$x + 1 = \pm 3$$
$$\boldsymbol{x = -4 \text{ or } 2}$$

Task 5

Tangents meet radii at 90°



Perpendicular to $y=6$ through $(10,6)$ is $\boxed{x=10}$

Line though $(0,6)$ and $(6,-2)$ has gradient $m = \frac{\Delta y}{\Delta x} = \frac{-8}{6} = -4/3$

Perpendicular has gradient $-1 \div -4/3$ and through $(6,-2)$ so $y = 3/4\,x + c$

$-2 = 3/4(6) + c$

$c = -2 - 18/4 = -13/2$

so $\boxed{y = 3/4\,x - 13/2}$

Centre is where $x=10$ and $y = 3/4\,x - 13/2$ cross.

We know $x=10$, $y = 3/4(10) - 13/2 = 1$.

Centre is at $(10,1)$

Task 6

There are 12 unique solutions to the problem:

#1:
18 x 1 x 2 = 36
2 x 14 x 7 = 196
5 x 15 x 3 = 225
2 x 8 = 16
6 x 3 x 2 x 9 = 324

#2:
18 x 1 x 2 = 36
7 x 14 x 2 = 196
5 x 15 x 3 = 225
2 x 8 = 16
6 x 3 x 2 x 4 = 144

#3:
18 x 1 x 2 = 36
7 x 14 x 8 = 784
5 x 15 x 3 = 225
2 x 2 = 4
6 x 3 x 2 x 9 = 324

#4:
18 x 1 x 2 = 36
8 x 14 x 7 = 784
5 x 15 x 3 = 225
2 x 2 = 4
6 x 4 x 2 x 3 = 144

#5:
18 x 4 x 2 = 144
7 x 14 x 2 = 196
5 x 15 x 3 = 225
2 x 8 = 16
6 x 3 x 2 x 1 = 36

#6:
18 x 4 x 2 = 144
7 x 14 x 2 = 196
5 x 15 x 3 = 225
2 x 8 = 16
6 x 9 x 2 x 3 = 324

#7:
18 x 4 x 2 = 144
7 x 14 x 8 = 784
5 x 15 x 3 = 225
2 x 2 = 4
6 x 3 x 2 x 1 = 36

#8:
18 x 4 x 2 = 144
8 x 14 x 7 = 784
5 x 15 x 3 = 225
2 x 2 = 4
6 x 9 x 2 x 3 = 324

#9:
18 x 9 x 2 = 324
7 x 14 x 2 = 196
5 x 15 x 3 = 225
2 x 8 = 16
6 x 3 x 2 x 1 = 36

#10:
18 x 9 x 2 = 324
7 x 14 x 2 = 196
5 x 15 x 3 = 225
2 x 8 = 16
6 x 4 x 2 x 3 = 144

#11:
18 x 9 x 2 = 324
7 x 14 x 8 = 784
5 x 15 x 3 = 225
2 x 2 = 4
6 x 3 x 2 x 1 = 36

#12:
18 x 9 x 2 = 324
7 x 14 x 8 = 784
5 x 15 x 3 = 225
2 x 2 = 4
6 x 3 x 2 x 4 = 144

Task 7

Ab-surd!

Solution

Fill in the blanks in the equivalent fractions and write the multiplier that gets you from one numerator to the next.

(1) $$\frac{1}{\sqrt{2}}\left(\times\frac{}{}\right)=\frac{\sqrt{2}}{}\left(\times\frac{}{}\right)=\frac{\sqrt{6}}{}\left(\times\frac{}{}\right)=\frac{}{6}$$

$$\frac{1}{\sqrt{2}}\left(\times\frac{\sqrt{2}}{\sqrt{2}}\right)=\frac{\sqrt{2}}{2}\left(\times\frac{\sqrt{3}}{\sqrt{3}}\right)=\frac{\sqrt{6}}{2\sqrt{3}}\left(\times\frac{\sqrt{3}}{\sqrt{3}}\right)=\frac{\sqrt{18}}{6}$$

What do you notice about the multipliers when going from a fraction with a surd in the denominator to one without?

(2) $$\frac{2}{5\sqrt{3}}\left(\times\frac{}{}\right)=\frac{}{15}\left(\times\frac{}{}\right)=\frac{2\sqrt{6}}{}\left(\times\frac{}{}\right)=\frac{}{60}$$

$$\frac{2}{5\sqrt{3}}\left(\times\frac{\sqrt{3}}{\sqrt{3}}\right)=\frac{2\sqrt{3}}{15}\left(\times\frac{\sqrt{2}}{\sqrt{2}}\right)=\frac{2\sqrt{6}}{15\sqrt{2}}\left(\times\frac{2\sqrt{2}}{2\sqrt{2}}\right)=\frac{8\sqrt{3}}{60}$$

The third to the fourth fraction might have been more challenging that the others. Did you find the multiplier first, or did you work out the numerator from one of the other fractions provided?

(3) $$\frac{5}{2+\sqrt{2}}\left(\times\frac{}{}\right)=\frac{10-5\sqrt{2}}{}\left(\times\frac{}{}\right)=\frac{}{20+10\sqrt{2}}$$

$$\frac{5}{2+\sqrt{2}}\left(\times\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)=\frac{10-5\sqrt{2}}{2}\left(\times\frac{10+5\sqrt{2}}{10+5\sqrt{2}}\right)=\frac{50}{20+10\sqrt{2}}$$

(4) $\dfrac{2-\sqrt{3}}{4}\left(\times\dfrac{\quad}{\quad}\right)=\dfrac{\quad}{8+4\sqrt{3}}\left(\times\dfrac{\quad}{\quad}\right)=\dfrac{\quad}{16}$

$$\frac{2-\sqrt{3}}{4}\left(\times\frac{2+\sqrt{3}}{2+\sqrt{3}}\right)=\frac{1}{8+4\sqrt{3}}\left(\times\frac{8-4\sqrt{3}}{8-4\sqrt{3}}\right)=\frac{8-4\sqrt{3}}{16}$$

Identify the rationalised fractions in the above lines. What do you notice about the multipliers when moving from a fraction with a surd (square root) in the denominator to a rationalised fraction?

Some examples from above include:

$$\frac{2}{5\sqrt{3}}\left(\times\frac{\sqrt{3}}{\sqrt{3}}\right)=\frac{2\sqrt{3}}{15},\quad \text{and} \quad \frac{2\sqrt{6}}{15\sqrt{2}}\left(\times\frac{2\sqrt{2}}{2\sqrt{2}}\right)=\frac{8\sqrt{3}}{60}$$

Can you see how to rationalise fractions with a single term in the denominator?

$$\frac{5}{2+\sqrt{2}}\left(\times\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)=\frac{10-5\sqrt{2}}{2},\quad \text{and} \quad \frac{1}{8+4\sqrt{3}}\left(\times\frac{8-4\sqrt{3}}{8-4\sqrt{3}}\right)=\frac{8-4\sqrt{3}}{16}$$

What is special about the denominators here?

How would you rationalise fractions in the following form: $\dfrac{a}{\sqrt{b}}$, $\dfrac{a}{b\sqrt{c}}$ and $\dfrac{a}{b+\sqrt{c}}$?

- The simplest way to rationalise $\dfrac{a}{\sqrt{b}}$ would be to multiply by $\dfrac{\sqrt{b}}{\sqrt{b}}$, giving $\dfrac{a\sqrt{b}}{b}$.
- For $\dfrac{a}{b\sqrt{c}}$ we could write: $\dfrac{a}{b\sqrt{c}}\times\dfrac{\sqrt{c}}{\sqrt{c}}=\dfrac{a\sqrt{c}}{bc}$.

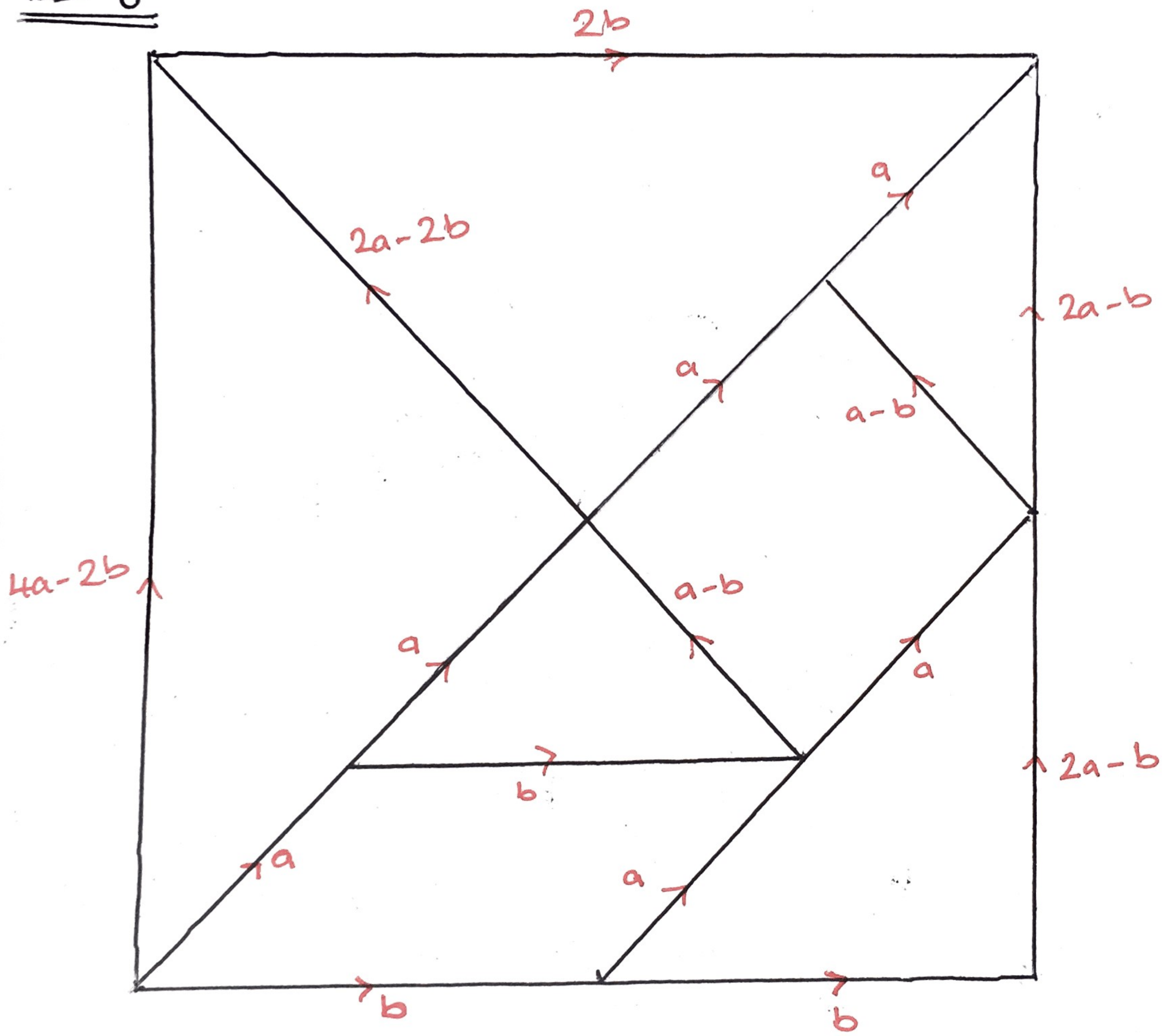
- For denominators that contain 2 terms such as $\dfrac{a}{b+\sqrt{c}}$, the examples above showed that multiplying by the difference of two squares would rationalise the denominator, as of the terms containing surds will cancel with each other: $\dfrac{a}{b+\sqrt{c}} \times \dfrac{b-\sqrt{c}}{b-\sqrt{c}} = \dfrac{ab-a\sqrt{c}}{b^2-c}$.

Is there more than one way to rationalise a fraction?

We have seen we can rationalise $\dfrac{2}{5\sqrt{3}}$ by multiplying by $\dfrac{\sqrt{3}}{\sqrt{3}}$. What happens if you try and rationalise by multiplying by $\dfrac{2\sqrt{3}}{2\sqrt{3}}$?

Does this idea work for fractions with two terms in the denominator? What different multipliers could you use to rationalise $\dfrac{5}{2+\sqrt{2}}$?

Task 8



Vectors Spider 2 - Answers

$\underline{\mathbf{p}} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\underline{\mathbf{q}} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

$3\underline{\mathbf{p}} - 4\underline{\mathbf{q}}$	$\begin{pmatrix} 6 \\ 13 \end{pmatrix}$
$3\underline{\mathbf{p}}$	$\begin{pmatrix} -6 \\ 9 \end{pmatrix}$
$4\underline{\mathbf{q}}$	$\begin{pmatrix} -12 \\ -4 \end{pmatrix}$
$4\underline{\mathbf{q}} - \underline{\mathbf{p}}$	$\begin{pmatrix} -10 \\ -7 \end{pmatrix}$
$\underline{\mathbf{p}} + \underline{\mathbf{q}}$	$\begin{pmatrix} -5 \\ 2 \end{pmatrix}$
$2\underline{\mathbf{p}} + 3\underline{\mathbf{q}}$	$\begin{pmatrix} -13 \\ 3 \end{pmatrix}$
$\underline{\mathbf{q}} - \underline{\mathbf{p}}$	$\begin{pmatrix} -1 \\ -4 \end{pmatrix}$
$\underline{\mathbf{p}} - \underline{\mathbf{q}}$	$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

Task 10

Review question R5281

Solution

From the inequalities

$$y - 2x > 0, \qquad x + y > 3, \qquad 2y - x < 5$$

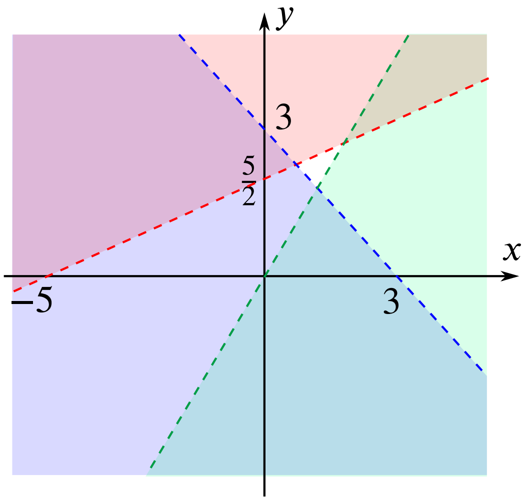
deduce that

$$\frac{1}{3} < x < \frac{5}{3}, \qquad 2 < y < \frac{10}{3}$$

Consider the three inequalities in the plane:

- $y - 2x = 0$ crosses the axes at $(0,0)$ and has gradient 2 (it rearranges to $y = 2x$). The region $y - 2x > 0$ lies above this line, as it includes the point $(0,1)$.
- $x + y = 3$ crosses the axes at $(0,3)$ and $(3,0)$. The region $x + y > 3$ lies above this line, as it does not include the point $(0,0)$.
- $2y - x = 5$ crosses the axes at $(0, \frac{5}{2})$ and $(-5,0)$. The region $2y - x < 5$ lies below this line, as it includes the point $(0,0)$.

We can therefore sketch these three regions in the plane. Here the shaded regions are those which **do not** satisfy the inequality.



We can see from our sketch that there is a small triangular region satisfied by all three inequalities. We determine where each pair of lines intersects in order to determine the coordinates of the vertices of this triangle.

$$\begin{aligned}
y = 2x \quad &\text{and} \quad x+y=3 && \implies && x=1, && y=2 \\
y = 2x \quad &\text{and} \quad 2y-x=5 && \implies && x=\tfrac{5}{3}, && y=\tfrac{10}{3} \\
x+y=3 \quad &\text{and} \quad 2y-x=5 && \implies && x=\tfrac{1}{3}, && y=\tfrac{8}{3}
\end{aligned}$$

From these we can deduce the minumum and maximum extent of both x and y:

$$\frac{1}{3} < x < \frac{5}{3}, \qquad 2 < y < \frac{10}{3},$$

as required.

... and hence that the given inequalities cannot be satisfied simultaneously by integral values of x and y.

The only integer value of x that satisfies $\frac{1}{3} < x < \frac{5}{3}$ is $x = 1$.

The only integer value of y that satisfies $2 < y < \frac{10}{3}$ is $y = 3$.

Does the point $(1,3)$ obey all three inequalities simultaneously?

$$\begin{aligned}
y-2x>0: &\quad 3-2=1>0 \quad \checkmark \\
x+y>3: &\quad 1+3=4>3 \quad \checkmark \\
2y-x<5: &\quad 6-1=5 \not< 5 \quad \text{X}
\end{aligned}$$

Thus there is no integer pair of x and y that satisfies all three inequalities simultaneously.

Task 11

Following the chain you should obtain the following solution:

"ANYTHING TO THE POWER ZERO IS 1"

Task 12

Start by calling the person in part **(ii)** Xander (X), the person in part **(iii)** Yolanda (Y), and the person in part **(iv)** Zak (Z). You can then put all the information from the first 5 parts of the question in a table:

Race	1st	2nd	3rd	4th	5th
There	Y		Z	A	X
Back	X	Z	Y	B	

You now know that Ahmed and Bachendri are **not** any of X, Y and Z. The missing space for the race to the tree must be Bachendri and the missing space on the race back must be Ahmed. Then use the information from the last two parts to work out which of X, Y and Z must be Charlie.

Final answer:

Race	1st	2nd	3rd	4th	5th
There	E	B	C	A	D
Back	D	C	E	B	A